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CERTAIN UNDEFINED ELEMENTS AND TACIT ASSUMPTIONS IN THE FIRST BOOK OF EUCLID'S ELEMENTS.*

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For twenty-one centuries the spirit of Euclid has reigned supreme over elementary geometry. For the greater part of this time the terms "Euclid's Elements" and "Geometry" have been all but synonymous. In no other science save theology has a single book held undisputed sway for unnumbered generations. Educational reformers have often pointed to this universal acceptance of Euclidean authority as an evidence of dead conservatism on the part of teachers. It is due rather to recognition of the unparalleled skill of the great Elementarist, whose text still stands as the best among the thousands which have been written on the subject, and whose methods are so simple and direct as to be well within the abilities of a beginner.

The beginner in the study of geometry, nevertheless, encounters certain difficulties in detail which must be considered in teaching the subject. These tend to minimize interest on the part of the student and to make the task of the teacher doubly hard. In consideration of this fact the writer wishes to submit the following pedagogical principle in mathematics: that *where any considerable number of normally intelligent students, under*

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varying conditions, find serious difficulty in the mastery of a certain detail, the error is less likely to be in the teaching than in the development of the subject itself. This principle, if sound, would indicate that Euclid made mistakes, or, at least, that there are errors in the Euclidean text as we have it. We may as well admit that such is the case, though they are few in number. In the effort to correct these we can scarcely do better than to follow Veblen's dictum: *to endeavor to rewrite the beginning of geometry as Euclid might have written it.* But this is a very high standard for the experimenter to set for himself, and he is much more likely to fail than to succeed.

So far as we know, Euclid nowhere states explicitly his purposes in formulating a science of space relations. A perusal of the theorems of Book I, however, seems to bring into clear relief the following aims on the part of the author:

1. To show that there is only one geometry.
2. To set a standard of logical procedure.
3. To set forth the fact that geometry begins with a limited number of preliminary statements, so simple in character as to be undeniable—"self-evident," as they are sometimes called.
4. To reduce the number of these preliminary statements to a minimum.
5. To separate the geometry of space in two dimensions from that of space in three dimensions.
6. To show from certain well-known configurations the existence of other configurations.

Editors of the Euclidean text for the use of schools have adhered to these aims. Authors of geometrics based upon Euclid have, as a rule, departed definitely from one or another of them. Probably the best known of the substitutes for Euclid's Elements is the geometry of Legendre, whose work has formed the basis of most of the American texts of the past century. Euclid, in pursuit of the sixth of the aims just mentioned, endeavors to account for every line and point demanded in a demonstration by a previous postulate or construction. Legendre, in establishing a new sequence, has added to the postulates two, to the effect that a line has a midpoint, and that an angle has a midray, both of which configurations he later constructs by processes which

depend for their validity upon certain theorems which employ the construction themselves. Legendre's American successors have as a rule gone further and introduced into their demonstrations any configuration which pleased them, thus departing entirely from Euclid's rule in the matter. This seems to be a serious pedagogical blunder, as well as a logical error which students are not slow to notice. To tell a student offhand that this or that line or point exists, without having proved its existence and shown how to establish it, is bad teaching and worse geometry. Recent texts in America have shown a tendency to return to Euclid in this particular.

The question before us is: Wherein, if at all, does Euclid himself fail in respect to the six aims referred to.

With regard to the first, it may as well be understood that he failed utterly. It is now known that there are many geometries based upon as many sets of assumptions. One of these, for historical reasons, is called Euclidean. The others have various names, which need not concern us here.

As to the second aim, Euclid's logic is unquestionably superior to that of any of his successors, up to the present period, which has been marked by a most careful examination into the processes of symbolic logic; the names of Hilbert, Pieri, Cantor, Dedekind and Russell come at once to mind, and in our own country those of Moore, Veblen, and Huntington. It is safe to say, however, that these mathematicians have set for themselves a general standard hardly higher than that of Euclid, but, instead, in the field of geometry, have been interested chiefly in correcting his mistakes.

Regarding the third aim: Euclid's axioms and postulates are all self-evident, except the one numbered twelve in the list given. About this one, the famous parallel postulate, an immense literature has been written. It looks as if it *ought* to be demonstrable—whence many sleepless hours and wearied intellects, and an abundant flood of ink, from the days of Proclus to the dreams of the latest neophyte. Its independence was fully established by Hilbert.

The real rub comes, however, with the fourth and fifth aims just mentioned. The latter is in a measure artificial, and many modern writers abandon it entirely. The present basis for pre-

serving the distinction between plane and solid geometry is that of convenience in arrangement of time units in teaching the subject. This has led to the rather ridiculous result that in many schools a highly theoretical plane geometry is required of all students, while an extremely practical solid geometry is elected by only a few.

As to a minimum list of assumptions, the idea came to prevail in the study of geometry that *nothing should be assumed which could be proved.* Whether Euclid had this idea we do not know. Certainly many of his theorems are "self-evident" enough; but he proves them with very great care. On the other hand there are many concepts and ideas, seemingly deserving of explicit statement, which he mentions not at all. I believe that this fact constitutes the great stumbling block to beginners in the study of Euclid. The *pons asinorum,* for example, is really of faulty construction, and the much-maligned beast, like his predecessor who bore Balaam, often shows keener powers of perception than his master.

A set of assumptions, to pass muster nowadays, must meet three requirements, designated as follows:

A. *Consistency,* meaning that they do not lead to contradiction.

B. *Independence,* meaning that no one of them can be deduced from the others.

C. *Categoricalness,* meaning (roughly) that they characterize the science for which they are devised, and no others.

Euclid's axioms and postulates do not meet these tests. A set of assumptions which will do so constitutes the absolute beginning of the subject. A number of such sets have been written, though the proofs that they meet the foregoing conditions are not complete. The set which is probably the best known in this country is that of Veblen, which is derived from projective geometry. To this set we will return presently.

Euclid's Elements have been edited, of course, many times. The best edition, one which should be available to every teacher of geometry, is that of Sir Thomas Heath (Cambridge, 1908). The references given below are to Todhunter's edition, long accepted as a standard. It can hardly be said of Todhunter, as may freely be asserted of many authors of geometries, that he has added to the number of logical errors which occur in Euclid.

Considering Todhunter's text, then, we may note first as to the *definitions:*

Numbers 1 and 2 do not define at all. The idea of length is much more complex than that of line. Number 4 defines "evenly" but not "straight line." Number 7 combines a definition with an assumption, neither of which is clearly stated. Numbers 8 and 9 define "inclination," and "different directions," but not "plane angle." Number 10 presupposes the existence of two lines under the given conditions, and also presupposes the existence of equal angles, a fact which Euclid is at great pains to prove in Book I, Prop. 23. Number 15 presupposes the existence of equal lines through one point. Of the other definitions it may well be that, if they were distributed at various points in the text, they would escape the error of involving "presuppositions." It is quite conceivable that Euclid so intended them to be used. Of the terms referred to, however, attempts in elementary texts at rigorous definition have only made a bad matter worse, as a rule.

As to the postulates:

The distinction between "Postulates" and "Axioms" is due not to Euclid, but to later editing. It is, of course, a fanciful one. Following the familiar usage, however, Postulate 1 should be combined with Axiom 10, to read "One and only one straight line can be drawn." Postulate 2 is generally superseded, now-a-days, by a distinction in definition between "line" and "line-segment." As it stands it is a play upon the word "straight." Axioms 1 to 7 overlook entirely the differences between the uses of the word "equal" when applied to lines, angles and areas. They really presuppose an assumption to the effect that each of these three things stands in direct one-to-one correspondence with the number system of Algebra. This is true, of course, but not "self-evident," as is shown by many familiar illustrations.

Axiom 9 is "axiomatic" enough to satisfy the most fastidious, but it is employed in numerous demonstrations without the slightest effort to *prove* which of two magnitudes is the whole and which is the part.

In axioms 11 and 12 Euclid is on sound foundations, provided

a process of "taking together" is defined. This he does not do, however.

If we now turn to the propositions:

Proposition 1 assumes that the two circles he describes will intersect. This seems obvious enough in this case, but so do a large number of other things which he, and we after him, are at great pains to prove; as in his own very next propositions, which the stomach of a present-day teacher of geometry revolts at, preferring to assume, without statement, that a compass will "stay put" when it is taken from the paper.

In proposition 4, however, Euclid throws caution to the winds and deliberately picks up a triangle as if it were a biscuit and deposits it upon another one. This operation has probably caused more annoyance to teachers and students than any other one thing in the beginning of geometry. I believe that every American text but one (Halstead) employs the same method of proof. One of my students a few days ago characterized it as "pure bunk."

Countless explanations are printed or offered orally. The first is to the effect that there is really an assumption of geometry which states that any figure can be moved from one place to another without altering the magnitude of its sides, its angles or its area. This assumption has high authority, but, as Russell points out, it presupposes motion as a geometric process, whereas motion itself requires to be defined in geometric terms. "Rigidity" of a geometric concept is about as conceivable as rigidity of a happy thought or of a devout hope. Pieri's list of assumptions for the foundation of geometry begins with motion. But this alters entirely the basis of the subject, and eliminates the necessity for a large number of propositions which we prove with assiduous regularity.

Proposition 5 now demands out attention. It is true that Euclid nowhere in Book I explicity states that all points are in the same plane. He presupposes it, however, in Proposition 30, and makes no reference to points outside of his fixed plane before Book XI. It is perfectly apparent, however, that the dictum (due, I think, to Pascal) of substituting the proof for the thing proved cannot be applied here. For to bring any pair of triangles from the figure into coincidence with each other

it is necessary not only to move one of them, but to *turn it bottom-side up,* which involves definitions of the top-side and the bottom-side of a plane triangle, and Heaven only knows what else, including space of three dimensions.

Propositions 6 and 7 taken together really constitute the proof of the third triangle congruence (three sides respectively equal); they are open to the same difficulty as the preceding. It is universally admitted that the familiar proof of this theorem is an improvement upon the method of Euclid. In 6, moreover, as Russell points out, Euclid first employs an axiom of which he is apparently wholly unconscious, though it is most essential to his sequence, viz.: that if AB, AC, and AD meet a line not through A in three points B, C, and D, then if B lies between C and D, then the angle BAC is less than the angle BAD; and in 7 there is need for a further assumption, that if the same three lines are cut by a fifth line in B', C', and D', then if C lies between B and D, C' must lie between B' and D'. In passing it may be noted that this axiom does not apply in elliptic space.

Proposition 12 employs without proof or explicit statement the fact that a straight line has two sides, and also that a circle cuts a straight line, if at all, in two and only two points.

Proposition 16 is open to the same objection as proposition 7. This is important when it is considered that it is upon this theorem that Euclid's whole theory of parallels rests.

Proposition 23 proves the existence of an angle when a given line at a given point equal to a given angle. It makes no mention of the important fact that there are two such. It should be noted, moreover, that without this proposition the student cannot construct the conditions laid down for Proposition 4. It looks very much as if Euclid had had as much trouble with Proposition 4 as the rest of us, and finally given the proof up as a bad job.

But enough of fault finding. Let us turn to our moral lessons. How should these and similar points of criticism effect our teaching of geometry?

The first lesson to be drawn is this: that no sequence in elementary geometry, not even that of Euclid, is sacred. On the other hand, even a hasty perusal of the fundamental principles of geometry as formulated by Hilbert, Pieri or Veblen, con-

vinces the reader that the statement of them involves a degree of abstraction far too profound for the mind of the beginner in the subject. In other words, geometry, like any other science, has its psychological as well as its logical origins, and these should be employed fully in the first presentation of the subject, instead of plunging the student unprepared into a sea of doubtful logic, as has been done in the past.

It is easy, however, to err in the other extreme of throwing logic entirely to the winds, and substituting empirical tests for every demonstration which seems a little difficult or unusual.

The better plan is to point out explicitly certain elements which are left undefined, and certain propositions which remain unproved, establishing their reasonableness by experiment, and to follow these up with accurate definitions of other terms and logical demonstrations of other theorems. In the choice of these undefined elements and unproved propositions, it may be well to follow Euclid rather closely. The "assumptions of order," should be assumed tacitly at the outset; they can be mentioned informally when needed. Clearly enough Euclid really employs many undefined elements. We have seen that many of his definitions are faulty. Others are too abstract to be grasped by the beginner. Certain of these geometric elements we should specify by name so as to employ them in defining other configurations:

Point	Curved surface
Straight line	Equality of lines
Curved line	Equality of angles
Plane	Equality of areas

Confusion only is the result when the three equalities mentioned are "defined" by superposition. It is by no means difficult for the beginner to grasp the contrary idea, that superposition is defined by equalities. This involves the omission of propositions 1, 2 and 3 from the Euclidian sequence.

Hilbert, as is well known, states Euclid I, 4, in part at least, as an assumption; the other two congruence theorems for plane triangles are easily derived from this, as Euclid shows, by indirect method. In general, it is probably the best procedure for beginners to regard all three as assumptions, and along with

them Euclid's two tacit assumptions regarding circles, referred to above. This leaves untouched, however, the difficulty which the student encounters with Euclid I, 5 (the *pons asinorum*), and with the introduction to the study of parallel lines.

Veblen's assumptions are fewer in number than those of Hilbert, and their significance is more easily grasped. As they constitute a satisfactory foundation for metrical geometry, it is desirable to note any necessary points of departure from them in elementary teaching.

Assumptions I to VI, in order to meet as closely as possible the criteria of independence, consistency and categoricalness, and for accuracy in definition, are so phrased that a beginner would hardly grasp their significance. For them may be substituted, in introducing the beginning to the subject, the familiar criteria for locating points on a straight line.

Assumption VII defines three-dimensional geometry, and Assumption VIII limits space to three dimensions at most. These belong to solid geometry. For plane geometry, VII would read:

All points are in one plane.

Considering now assumption IX, which, as Legendre shows, is the equivalent of Euclid's "parallel postulate." Hilbert makes this assumption read: ". . . there is *one and only* one line coplanar with *a* and not meeting *a*." Let us accept this form in spite of Euclid I: 27. Suppose now that "parallel lines" and "parallelogram" are defined in the usual fashion, and that there is added this assumption:

Opposite sides of a parallelogram are equal.

If now to equal and parallel line-segments we apply assumptions X to XIII, we have the foundations of a geometry independent of "rigid motion," and involving a large number of familiar propositions, which are demonstrable with Euclidean exactitude, but by methods which are simpler than those usually employed. Among these are:

Two equal and parallel lines determine a parallelogram.

To double a straight line.

To bisect a straight line.

The midline of a triangle is parallel to the base and equal to one half of it.

To divide a line into any number of equal parts.

The diagonals of a parallelogram bisect each other.

In this development the familiar "subtraction, multiplication, and division axioms," as applied to parallel straight lines, may be stated as assumptions, though they are in reality theorems.

The definition of "angle" now demands our attention. Euclid's definition, as has been said, is unsatisfactory. But if the familiar definition as "the figure formed by two rays having a common origin" is accepted, with the usage of "straight angle" (an excellent idea with an unhappy name), then Euclid I: 13, 14, 15, and certain of the familiar definitions of various kinds of angles follow at once. The sum and the difference of two angles are also definable, when the angles have a common vertex.

Assumption XIV in Veblen's set serves, as he has shown, to establish equality of angles generally, and to replace Hilbert's axiom regarding congruent triangles. As a partial substitute for this assumption, let us state the following:

Corresponding (exterior-interior) angles of parallel lines are equal.

This assumption opens the door for the usual sequence regarding angles and parallel lines, including the proof for the sum of the angles of a triangle.

We may call the geometry thus obtained the GEOMETRY OF TRANSLATION. (It also involves in some measure the idea of point-reflexion, but this is hardly essential to the sequence.)

The geometry of translation does not permit of the equality of lines which are not parallel. This concept is introduced by Euclid's definition of the circle. One of Euclid's tacit assumptions may now be stated explicitly, to the effect that:

A straight line may have not more than two points in common with a circle. And this should be followed by Veblen's assumption XV.

Let us now add arbitrarily this assumption:

When two circles intersect, the line-segments joining their points of intersection to any given point on the line of centers are equal. These line-segments are called *symmetric* segments, and their corresponding extremities *symmetric* points.

It will be noticed at once that this assumption states in geometric form a fact which is illustrated in paper folding. The geometry of paper folding has been developed with considerable

detail. It is necessary, however, from a purely geometrical point of view, to account for the physical phenomena of paper folding in terms of formal definitions, assumptions, and propositions. *Symmetric rays* and *symmetric lines,* in particular are readily defined.

The analogous assumption regarding angles in the GEOMETRY OF SYMMETRY appears to be this:

An angle between two rays is equal to the angle between their symmetric rays. This assumption includes as a limiting case one to the effect that *symmetric rays make equal angles with the axis.*

It is now assumed that the axioms of equality extend to equal lines and equal angles as thus defined.

This geometry has been developed at some length in many texts. On the basis above given it is independent of parallelism, and is applicable to hyperbolic space, or to elliptic space when necessary restrictions are placed upon the radii of the intersection circles. Symmetry also defines the right angle, and justifies most of the familiar constructions of geometry. The equality of the base angles of an isosceles triangle is a particular case of symmetry. It cannot be proved in Euclidean fashion without some form of "folding over."

The theorem "A line perpendicular to a radius of a circle at its extremity is tangent to the circle" is also proved by symmetry, as are also the theorems "The bisectors of the angles of two intersecting lines are the locus of the centers of circles tangent to the two lines," and "The perpendicular bisector of a line joining two given points is the locus of the centers of circles passing through the two points."

The principles of symmetry, when properly formulated, also hold for spherical geometry as that subject is usually defined. It is to be noted merely that the physical phenomenon of paper folding is not applicable here.

The third of the introductory steps to geometry has to do with *rotation*. If motion is employed for illustration only, and not for demonstration, certain assumptions are needed, which can be stated most clearly by adding one to two new terms to the vocabulary of the subject.

Two points are said to be *concentric* about a third point when

they lie on a circle of which the third point is the center. Two lines are said to be *concentric* about a point if they are tangent to a circle of which the point is the center. Two rays are *concentric* if their vertices are concentric and the lines of which they form a part are concentric about the same point, and if either they both do or both do not include the points of tangency.

Two segments are *concentric* if they are laid off on concentric rays and their extremities are concentric about the same point as are the rays.

Two angles are *concentric* if their sides and vertices respectively are concentric about the same point, and if either they both do or both do not contain that point.

From these definitions and the locus theorems referred to above it is easy to construct the center for two given non-parallel rays. In so doing it will be found that in general *two* locus intersections are obtained. The one should be chosen which is or is not separated from the vertex by the line through the origin of the rays, according as the rays themselves are or are not so separated.

The two assumptions of this geometry can now be stated:

Concentric segments are equal, and

Concentric angles are equal.

To these may be added the familiar proposition:

In the same circle, equal chords subtend equal central angles.

This means in terms of motion that of two such segments one can be rotated about their common center to coincidence with the other, and that of two such angles one can be rotated about their common center either to coincidence with or to a position symmetrically adjacent to that of the other.

The theorems, constructions and special instances of this Geometry of Rotation are for the most part unfamiliar, but they are relatively simple and very interesting.

We now come to the subject of congruent triangles.

It appears that if in the triangles ABC and DEF, located at random, AB equals DE by the principles of the geometry of translation, or of that of symmetry, or of that of rotation, it is possible to construct on the line DE a triangle $A'B'C'$ whose sides and angles will be respectively equal to those of the triangle ABC. This triangle is said to be *congruent* to the triangle

ABC, the term "congruent," in this instance, having the same meaning as Euclid's "equal in all respects." If on the line *DE* there is already a triangle *DEF* such that $DF = A'C'$ and $EF = B'C'$ then the angles of triangle $A'B'C'$ will either coincide with or be symmetrical to those of triangle *DEF*; for otherwise two circles would have three distinct points in common.

It is customary in plane geometry to call symmetric triangles congruent. Euclid does this tacitly in Proposition I, 5, as has been seen. If this is done explicitly, we have proved the third congruence theorem in its usual form.

The other two congruence theorems are established in similar fashion.

The foregoing outline for an introduction to geometry is based, as is apparent, upon the substitution of eight or nine assumptions in place of Veblen's assumptions IX and XIV.

From the point of view of reducing the number of assumptions to a minimum this is a palpable surrender of principle. But it is now fully understood that many, if not most, of the assumptions of metrical geometry are in reality demonstrable propositions of projective geometry, and that the Euclidean geometry of our forefathers is really one of many "subgeometries" of projective geometry, namely, the parabolic metric geometry. To add to the list of assumptions of Euclidean geometry, then, is merely to make a new selection from the theorems of projective geometry, and the above list is suggested as an experiment in this direction.

Of the pedagogic advantages which this procedure seems to suggest the following may be noted:

1. An analysis of motion is included which is closely related to the study of kinematics.

2. The methods of construction are brought into closer relation with those which prevail in mechanical drawing.

3. The principles as laid down appear first as direct inferences from processes of drawing, the instruments being the straight edge, the ruler and some form of parallel-line machine.

4. Many demonstrations are reduced to much simpler terms. There is need at this time for closer attention to what may be called the "psychology of logic"—the nature of a definition, and the offence of unnecessary circumlocution.

5. Vector analysis has a well-defined point of contact with elementary mathematics.

In short a channel is opened for a course in inventional geometry which can later be brought into harmony with the deductive science on either a relatively or a purely formal basis. And the "axiom of parallels" thus appears in its proper perspective rather than as a miraculous interference with the processes of human logic.

For convenience of reference the following captions taken from Todhunter's Edition of Euclid and from Veblen's Projective Geometry (Vol. II) are subjoined.

Euclid's Elements.

Definitions.

1. A point is that which has no parts, or which has no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

5. A superficies is that which has only length and breadth.

6. The extremities of a superficies are lines.

7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

10. When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

11. An obtuse angle is that which is greater than a right angle.

12. An acute angle is that which is less than a right angle.

13. A term or boundary is the extremity of any thing.

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the center of the circle.

17. A diameter of a circle is a straight line drawn through the center, and terminated both ways by the circumference.

(A radius of a circle is a straight line drawn from the center to the circumference.)

Postulates.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any center, at any distance from that center.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Propositions.

1. To describe an equilateral triangle on a given finite straight line.

2. From a given point to draw a straight line equal to a given straight line.

3. From the greater of two given straight lines to cut off a part equal to the less.

4. If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

5. The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced the angles on the other side of the base shall be equal to one another.

6. If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

7. On the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.

8. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle which is contained by the two sides, equal to them, of the other.

9. To bisect a given rectilineal angle, that is to divide it into two equal angels.

10. To bisect a given finite straight line, that is to divide it into two equal parts.

11. To draw a straight line at right angles to a given straight line, from a given point in the same.

12. To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

13. The angles which one straight line makes with another straight line on one side of it, either are two right angles, or are together equal to two right angles.

14. If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

15. If two straight lines cut one another, the vertical, or opposite, angles shall be equal.

16. If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

17. Any two angles of a triangle are together less than two right angles.

18. The greater side of every triangle has the greater angle opposite to it.

19. The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

20. Any two sides of a triangle are together greater than the third side.

21. If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

22. To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

23. At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

24. If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides, equal to them, of the other, the base of that which has the greater angle shall be greater than the base of the other.

25. If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

26. If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to equal angles in each, then shall the other

sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

27. If a straight line falling on two other straight lines, make the alternate angles equal to one another, the two straight lines shall be parallel to one another.

28. If a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side equal to two right angles, the two straight lines shall be parallel to one another.

29. If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side; and also the two interior angles on the same side together equal to two right angles.

30. Straight lines which are parallel to the same straight line are parallel to each other.

31. To draw a straight line through a given point parallel to a given straight line.

32. If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

33. The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Veblen's Assumptions.

1. If points A, B, C are in the order $\{ABC\}$ they are distinct.

2. If points A, B, C are in the order $\{ABC\}$ they are not in the order $\{BCA\}$.

3. If points C and D $(C = D)$ are on the line AB, then A is on the line CD.

4. If A and B are two distinct points there exists a point C such that A, B and C are in the order $\{ABC\}$.

5. If three distinct points A, B and C do not lie on the same line and D and E are two points in the orders $\{BCD\}$ and (CEA), then a point F exists in the order $\{AFB\}$ and such that D, E and F lie on the same line.

6. There exist three distinct points, $A, B, C,$ not in any of the orders $\{ABC\}$, $\{BCA\}$, $\{CAB\}$.

7. If A, B, C are three non-collinear points, there exists a point D not in the same plane with A, B and C.

8. Two planes which have one point in common have two points in common.

9. If A is any point and a any line not passing through A, there is not more than one line through A coplanar with a and not meeting a.

10. If $A = B$, then on any ray whose origin is C there exists one and only one point D such that (A, B) is congruent to (C, D).

11. If (A, B) is congruent to (C, D) and (C, D) is congruent to (E, F) then (A, B) is congruent to (E, F).

12. If (A, B) is congruent to (A', B') and (B, C) is congruent to (B', C') and $\{ABC\}$ and $\{A'B'C'\}$, then (A, C) is congruent to (A', C').

13. (A, B) is congruent to (B, A).

14. If A, B, C are three non-collinear points and D is a point in the order $\{BCD\}$, and if $A'B'C'$ are three non-collinear points and D' is a point in the order $\{B'C'D'\}$ such that the point-pairs (A, B), (B, C), (C, A), (B, D) are respectively congruent to (A', B'), (B', C'), (C', A'), (B', D') then (A, D) is congruent to (A', D').

15. If the line joining the centers of two coplanar circles meets them in pairs of points P_1Q_1 and P_2Q_2 respectively such that $\{P_1P_2Q_1\}$ and $\{P_1Q_1Q_2\}$ the circles have two points in common, one on each side of the line joining the centers.

16. If $A, B, C,$ are three points in the order $\{ABC\}$ and B_1, B_2, B_3 are points in the order $\{ABB_1\}$, $\{AB_1B_2\}$ . . . such that (A, B) is congruent to each of the point-pairs (BB_1), (B_1B_2) . . . , then there are not more than a finite number of points B_1, B_2 . . . between A and C.

Suggested Changes and Additions for Elementary Instruction.

9. ". . . there is one and only one line. . . ."

Let "equal" be substituted for "congruent" as applied to lines and angles.

10*a*. If the rays (a, b) form an angle, and c is a third ray, then with c there are two and only two angles each equal to the angle (a, b).

11*a*, 12*a*, 13*a*, involve applications of the principles of equality to angles.

14. To be omitted.

T1. Opposite sides of a parallelogram are equal.

T2. Corresponding angles of parallel lines are equal.

S1. "Symmetric" segments are equal.

S2. "Symmetric" angles are equal.

R1. "Concentric" segments are equal.

R2. "Concentric" angles are equal.

R3. In the same circle equal chords subtend equal central angles.

Central High School,
Newark, N. J.

ASSOCIATION OF MATHEMATICS TEACHERS OF NEW JERSEY. REPORT OF THE COMMITTEE OF FIRST-YEAR HIGH-SCHOOL MATHEMATICS.

OUTLINE OF REPORT.

I. THE WORK OF THE COMMITTEE.

The earliest work of the committee was the preparation of a list of topics to be recommended for omission from first-year high-school algebra. The list was endorsed by vote of the Association at the December, 1916, meeting, and received the general assent of those responding to our questionnaire. It is printed

again here with the hope that it may accomplish good even in places where the constructive program of the committee may not be approved. The list points the way to a desirable simplification of first-year algebra, time being gained for thorough work on really essential topics, and for the somewhat broader foundation in mathematics.

With this list as a starting point, the committee prepared a fairly comprehensive questionnaire, which was mailed to all the members of the Association and to several authors of text-books. The committee wishes again to express its grateful appreciation of the many painstaking replies. A report concerning them was made at the spring meeting in 1917. Since that time the committee has been at work on a syllabus, and mention of progress has been made at each meeting. At various times Mr. Belcher, Mr. Strader, Superintendent Stevens, and Mr. Lord has addressed the Association, in each case reflecting upon some phase of the committee's problem.

Although we were appointed only to consider the first-year of the four-year non-technical high school, the general argument of our report as well as many of the details of the syllabus should be of help to those who are planning courses or who are teaching in technical or in junior high schools. It is true that a bit of algebra—the easiest equations and formulas, with some graph work—should be used somewhere in every seventh and eighth grade course in arithmetic. The systematic study of algebra must be taken up, however, at some point in the eighth or ninth year, and may then, in whatever type of school, be based largely upon our plan.

It should be stated that the recommendations of our report would be much more radical than they are if we felt that the average high school were ready for them. With our report available as a basis for teaching and for discussion, greater strides ahead will be made after a few years by a future committee.

II. Suggested Omissions from First-Year Mathematics.

(See note.)

1. Parentheses more complicated than one within the other.

2. Multiplication and division of polynomials having literal exponents or fractional coefficients. Use of detached coefficients.
3. Long division with divisors other than binomials.
4. Special rule for squares of polynomials.
5. Cases in factoring:
   (*a*) sum or difference of two cubes.
   (*b*) sum or difference of any two like powers, except the difference of two squares.
   (*c*) trinomials such as $x^4 + x^2y^2 + y^4$.
   (*d*) "factoring" which leads to irrational or imaginary factors.
6. Highest common factor and lowest common multiple as separate topics (the latter is unnecessary if pupils are drilled on changing to l. c. d.).
7. Change of signs in binomial denominators of fractions to be added.
8. Complex fractions.
9. All special tricks or devices such as those in fractional equations.
10. Transformation of proportions by "alternation," etc.
11. In simultaneous equation, the method of eliminating by comparison and the solution of examples with more than three unknowns.
12. Involution.
13. Square root of polynomials having fractional coefficients. Square root of a binomial surd.
14. Radicals of order higher than the second. Rationalization of binomial denominators. Radical equations.
15. Theory of exponents, except sufficient work with monomials to make clear the meaning of fractional and negative exponents.
16. In quadratic equations the "Hindu" method of completing the square.
17. Imaginary numbers, except for brief mention when encountered in the quadratic formula.
18. Equations in quadratic form. Simultaneous quadratics, except when one equation is linear and the example can easily be solved by substitution.

19. Theory of quadratics.

Note: Under the present college entrance requirements most of the topics listed above would find their place in the intermediate algebra course, a summary of which forms the concluding page of this report.

III. Simplification.

As a cursory examination of the foregoing list serves to show, a fundamental consideration has been that first-year work should not comprise the intensive study and mastery of half of the topics of algebra (*e. g.*, Algebra A1 of the College Entrance Examination Board), but should be an elementary study of a wider range of topics. The great need of simple and carefully graded subject matter strongly impresses most high-school mathematics teachers to-day, whether they consider the extremely youthful and untrained persons who compose their classes on the one hand, or the complexities and intricacies of the average textbook on the other.

Mr. Lord has carefully listed for the committee the new principles met with in the most widely used first-year algebra text: as finds the surprising total of ninety-seven, too long a list to be given here. One hardly realizes until face to face with such a list the extraordinary number of things a pupil has to get and keep in mind, and in view of this the great importance of not having too many elements in a problem. It is also true that not only are there many steps in the processes which might be added to his list but the pupil usually has to arrange the steps in an absolutely definite sequence. He must perform each step accurately or he will be lost in complications and will conclude that he does not know how to solve the problem. He has not had the experience to know where to dig out a mistake.

Although a list made out from our syllabus as nearly as possible on the same basis contains 78 principles instead of 97, the numbers are not a true index of comparative simplicity, because examples in the book are in most of the chapters more involved than those recommended by us.

A search through textbooks of college mathematics shows that the average problem does not contain as many or as difficult manipulations as are often demanded in high-school texts. The

most frequent process appears to be substitution in formula, and after the substitution is made the subsequent manipulation is not usually very involved. Increased ability in these mechanical operations will come automatically with the mental growth and mathematical experience of the pupil.

IV. Fundamental Plan.

The time often spent in the first year on these difficult manipulations, and on the topics now suggested for omission, could well be used for incidental excursions into neighboring fields of arithmetic and geometry, and for practice on such fundamentals as substitution of numbers (positive, negative, and fractional), the order of operations, graphs, checks, and mental drill on easy fractions and equations. Here should be mentioned, too, the translation of verbal statements into equations or formulas and vice versa, for if there is one thing our beginners lack it is the ability to read and plan intelligently.

V. Correlations.

A strong argument for planning a broader foundation for mathematics and for living than the traditional algebra alone is that we are not teaching applicants for college but fourteen-year-old boys and girls, half of whom will never graduate from school. Is it fitting not to afford them a glimpse now and then of the many-sided domain of mathematics, and some realization of the sweep and power of its laws and its language?

In the arithmetic which stands out frequently in this syllabus there is a three-fold purpose:

1. To keep alive a feeling of acquaintance with the arithmetic of the lower grades.

2. To make clear by the algebraic formula or equation many of the processes or problems of arithmetic, and

3. To gain impetus for the operations of algebra by showing their close analogy with those of arithmetic.

The topic from geometry in the first place afford the best available applications of formulas, whether as numerical substitution exercises, as literal equations, or as concise symbolic expressions of verbal rules. After paper-cutting and folding, measuring, or developmental questions have led to a new rule

the pupils should be asked each time to translate the rule into a formula. In the second place without the introduction of illustrations from scale-drawing and geometry our study of the important subject of ratio and proportion often fails to get anywhere, unless valuable time is taken to teach a bit of science or mechanics.

This chain of correlated ideas links up with the arithmetic of the grades and with the geometry of the following year. It is not to be forgotten that mathematics affords within itself rich territory for profitable correlations and cross-references.

VI. Textbook.

The supplementary work outside the usual textbook may sometimes prove so suggestive of additional illustrations and of class-room discussions that a word of caution may be in order. It is a commonplace that as teachers we should not permit ourselves to be enslaved by a text. At the same time any text presents a body of matter arranged with more or less skill by the author, and we must caution ourselves not to depart from the text without having carefully organized our material and our lesson-plans. Unless we can teach with confident mastery and unless we can avoid frittering away over-much time in the mere dictation of home-work problems it is necessary to follow a text closely. Otherwise pupils and teacher alike may become confused, and the satisfaction that comes from definite progress and accomplishment will be lacking.

VII. Teaching.

Particularly since the main field of interest lies in his own subject, the teacher of mathematics must make great effort to come to a complete understanding of his pupils. We must both foster and sustain their interest. It is happily true that most pupils like their algebra, and like to "do" examples, if they can secure results which they can check or can feel to be correct. How important it is then, to plan reasonable exercises for them and not invent stumbling blocks to put in their way! How important, also, to guide pupils sympathetically in correct habits of study, and to assign lessons intelligently! Conversation with a class about the difficulties they will encounter and suggestive

questions with and without spoken replies will supplement a preparatory rapid-fire oral review, a snappy quiz with brief written answers, or thoughtful perspective questioning. A well-assigned lesson is a lesson half won.

VIII. The Bright Pupil.

One is reminded that many of our pupils will be ready for the next forward step in half the time needed by others. The problem of the bright and the slow pupil is ever pressing in algebra, a subject needing such a great amount of detailed drills and reviews. The core of our first year's work is skill in the simpler algebraic processes, the "tool" operations, and in this mechanical work all pupils must attain efficiency. The time gained by the rapid pupils may best be used neither to prolong the drill work unnecessarily nor to advance to new topics ahead of the class but for intensive challenging exercises with each operation, and for additional problems of application or interpretation based on the operations already learned. No matter how necessary it may be to arrange simple and easily digested material for the class as whole the more capable ones require something to "chew on." The occasional propounding of extra credit problems goes far to meet this need and is reported as helpful by many teachers. With a certain essential minimum of formal skill, in which phase of the work the pupils of a class may advance together no pains should be spared to give the abler ones every possible opportunity and encouragement for work in thought problems, and in supplementary applications sometimes of a rather unconventional type. In short let us make sure that our strongest pupils are kept on their mettle to think.

IX. Syllabus of First-year Mathematics.

1. *Formulas and Numerical Substitution.*

(*a*) A letter representing a number. Meaning of square and cube of a number illustrated by area of square and volume of cube. Pupil may tabulate a list of squares and cubes.

(*b*) Meaning of *ab*. Area and perimeter of rectangle. Circumference and area of circle. Compasses: the construction

of a circle with a given radius and given center. Volume of a sphere, $V = \frac{1}{6}\pi D^3$.

(c) Translation of rules into formulas. Evaluations of algebraic expressions. Order of operations. Miscellaneous formulas including perhaps board measure $B = \frac{1bt}{12}$, falling body $s = \frac{1}{2}gt^2$ and $s = vt + \frac{1}{2}gt^2$, and H. P. of automobile $= \frac{ND^2}{2.5}$.

Note 1: The Commutative Law ($ab = ba$, $a + b = b + a$) and the Association Law ($a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$) may well be explained under (c). Illustrate by inquiring whether $A = bh$ is different from $A = hb$, why the factors of any term may be written in alphabetic order, or the best plan for obtaining the sum of $2\frac{5}{6} + 3\frac{4}{7} + 4\frac{1}{6}$ or the product of $97 \times 4 \times 25$.

Note 2: From the beginning the pupil should be taught to form the habit of obtaining a rough check on his results by making an estimate of the answer in round numbers, or by asking himself "Is my result a reasonable answer to this problem?"

2. *Equations and Problems.*

(a) Equations of the type $ax^n = b$.

(b) Translations of verbal statements into equations.

(c) Problems, more easily and more systematically solved by algebra than by arithmetic. Finding one dimension of a rectangle given the area and other dimension. Finding the diameter or the radius of a circle given the circumference (by solving an equation like $3.14D = 12''$).

3. *Graphs and Negative Numbers.*

(a) Statistical graphs: school enrollment, growth of population, crops, or monthly profits and expenses.

(b) Graphs of temperatures including readings below zero. Profile sections showing elevations above and below grade or water level. Simple questions, related to the graphs, involving negative and positive readings, increases and decreases, etc.

(*c*) Further illustrations of negative number: longitude and latitude, debts and assets, scoring in games, etc.

4. *Addition and Subtraction.*

(*a*) Addition. Bridge over from the addition of arithmetic by practice with denominate numbers. Besides the arrangement in columns emphasizes the collecting of like terms in such expressions as

$$5x^2 - 7x + 2 - x^2 + 2x - 1 - 5x + x^2.$$

Horizontal addition and tabulation in arithmetic may be reviewed. Check all addition by numerical substitution, for the most part by replacing each letter by unity.

(*b*) Subtraction. Making change; find what number added to the subtrahend would give the minuend. Check as in (*a*) and by addition.

Note 1: Difficulties in subtraction are best explained by reference to graph work.

Note 2: When doubt arise the numerical check is convincing, for instance: "How can -4 be subtracted from $+7$ and a difference $+11$ be obtained, greater than either of the numbers?" Let us answer by investing the difference between $12a + 7$ and $7a - 4$, checking with $a = 1$:

Example:	$12a + 7$	$12 + 7 = 19$	
	$7a - 4$	$7 - 4 = \underline{3}$	Check, letting $a = 1$.
		16	
	$5a + ?$	$5 + ? = 16$	

Are you sure of the values 19 and 3, and that their difference is 16? Then, if that is true, the answer $5a + 11$ is correct, for any second term other than $+11$ would fail to make 16.

(*c*) Parentheses. Signs of aggregation. Concentrate on expressions of common occurrence such as $(a - (b - c))$ or $((x + \frac{3}{4}y) - (x - y))$, giving few more complicated than these.

Insertion of parentheses, *e.g.*, to enclose the last three terms of $a^2 - b^2 + 2bc - c^2$ in a parentheses preceded by a minus sign. Check by numerical substitution.

Note: It is feasible to postpone all study of parentheses until

after multiplication, the minus sign before a parentheses being taken to indicate multiplication by minus one.

(d) Equations. (The following types in order (a, b, and c) representing arithmetical numbers):

(1) The notation of an equation as a balanced pair of scales. Subtracting the same quantity from each member. Examine such obvious arithmetical equalities as \$7 + \$3 = \$10, \$3 = \$10 — \$7.

(2) $ax + b = c$.

(3) $x - b = c$ and $ax - b = c$. "Making up a shortage" axioms used informally. Law of transposition discovered.

(4) Equation with parenthesis.

(5) A few problems, very simple in character.

5. *Multiplication and Division.*

(a) *Multiplication.*—Compare $5(3f + 2i)$ with 5 (3 ft. + 2 in.). Explain such examples as $3(2a - 5)$ by repeated addition,

$$\begin{array}{r} 2a - 5 \\ 2a - 5 \\ 2a - 5 \\ \hline 6a - 5 \end{array}$$

Thus getting the law that the product of two factors with unlike signs is negative. The law concerning like signs may be derived by multiplication $(3x - 2)$ by -5 or by $2x - 5$ and seeing which of the two possibilities for the final term $+10$ or -10 would satisfy a numerical check with $x = 4$.

In checking multiplication numbers greater than unity should usually be substituted, for with unity a mistake in exponents would not be revealed. The work may be limited to binomial multipliers and the product of two trinomials.

(b) *Special Cases of Multiplication.*—Type products $(a + b)^2$, $(a + b)(a - b)$, $(x + 5)(x + 2)$, $(3x + 4)(2x + 5)$. Rules for the first two types may well be committed to memory and the geometrical diagram for $(a + b)^2$ should be given. Squares of numbers such as 41, 65, and $5\frac{1}{2}$ afford interesting applications of the rule.

Note: $(n+\frac{1}{2})^2 = n(n+1)+\frac{1}{4}$ yields the well-known short method for the last two numbers.

By the aid of a few leading questions simple and brief exercises in factoring may follow each type product. There are some objections, however, to a systematic study of factoring at this point, before equations and problems have received the early attention merited.

(*c*) *Parentheses with Multiplications.*—Examples to contain additional practice with type-products. Examples such as $(a+b)^2(a+b)$ or $(3a+x)^3$, should not be omitted. The latter example is, of course, to be worked by ordinary multiplication, as the rule for the cube of a binomial is not studied.

(*d*) *Division.*—To find what number multiplied by the divisor would give the dividend. Limit the work to binomial divisors, excepting for two or three trinomials such as $(a^2+2am+m^2-k^2)\div(a+m-k)$. Check by multiplication and by numerical substitution, taking care to avoid a zero value for the divisor.

A little work should be given emphasizing remainders:

1. Show that x^3+y^3 is divisible by $x+y$.
2. Show that there is a remainder when x^2+y^2 is divided by $x+y$.
3. Show that the remainder when x^3-y^3 is divided by $x+y$ is $-2y^3$.
4. What is the remainder when ——— is divided by ———?

(Postpone the sum of difference of two cubes as a special case until the factor theorem is studied in intermediate algebra, but along with examples similar to those here stated, passing mention may be made of the type.)

Note: An admirable exercise when first teaching multiplication or division is to place an arithmetical example along side a corresponding one in algebra, and compare figure by figure, *e.g.*, divide 672 by 32 and $6t^2+7t+2$ by $3t+2$.

6. *Equations and Problems.*

(*a*) Simple equations now more thoroughly studied. Equations involving multiplication. Checks.

(*b*) Symbolic representation, especially to represent with one unknown letter two numbers given their sum, difference, product

or quotient, disguised in various wordings. Relate the algebraic representation to such arithmetical questions as "If the difference of two numbers is 6 and the smaller number is 17, what is the larger?"

(*c*) Translation of verbal statements into algebraic equations, etc. Illustrations from arithmetic will always help. "State in the form of an equation that the sum of 13 and 25 is 38" or that "seven exceeds four by three," etc. Areas and perimeters of squares and rectangles, etc., *e.g.,* "Write a formula for the perimeter, p, of a square in terms of its side;" "Write a formula for the number of feet, f, in the length of any line in terms of the number of inches, i."

(*d*) One-statement problems. One unknown number.

(*e*) Two-statement problems. One statement to be used to express one of the required numbers in terms of the other, the remaining statement being translated into an equation. To make clear the method do the same problem two ways.

(*f*) More complicated problems: number, money, uniform motion areas, etc. Some problems leading to identities such as "Show that the difference between the squares of two consecutive numbers is equal to the sum of the numbers."

7. *Factoring.*

The purpose of factoring explained. When is an expression in factored form? Illustrations. Note that factoring is of use in simplifying many mathematical computations, if care is taken to factor as much as possible before performing the calculations (*e.g.,* under the first and fifth types below).

1. Type form $ax + bx$. Common Monomial Factor. Highest Common Factor.

The common factor sought is the highest common factor, which should be explained with examples from arithmetic.

Perhaps this is the most important case in factoring. Examples similar to the following may be given, either with or without concrete explanation:

$$IID + IID,\ b + br,\ p + prt,\ vt - \tfrac{1}{2}at^2 = t(\ ?),$$

$$\frac{an}{2} + \frac{ln}{2} = \frac{n}{2} \quad (\ ?),$$

$$100h + 10t + u = h + t + u + 9(\ ?).$$

Brief studies may be made of:

(*a*) Perimeters, $P = 2l + 2w = -$? Construction of rectangles with compass.

(*b*) Area of triangle. Paper cutting and folding will best explain this, as it will (*c*) and (*d*). Construction of rectangles with altitudes.

(*c*) Area of trapezoid, $A = \frac{1}{2}bh + \frac{1}{2}b_2h = ?$

(*d*) Surface of cylinder,

$$S = 2\pi Rh \qquad E = 2\pi R^2 =$$
$$T = 2\pi Rh + 2\pi R^2 = ?$$

2. Type form $a^2 - 2ab + b^2$. Perfect trinomial square. Expressions which are not perfect squares should be scattered through the assignments, for otherwise the work of the pupil is likely to be superficial.

Supplying a missing term in order to complete a square:

$$x^2 + 10x + ?,\ 4a^2 - ? + 49b^2.$$

3. Type form $x^2 + px + q$. Simple quadratic trinomial.

4. Type form $ax^2 + bx + c$. General quadratic trinomial. Oftener called the "cross-products" case because of the method most successfully taught. (Schultze: "The Teaching of Mathematics in Secondary Schools.")

It should be emphasized at this point that type forms No. 2 and No. 3 are merely special cases of No. 4.

5. Type form $a^2 - b^2$. Difference of two squares. The economy in computation which may be accomplished by factoring is well shown by examples concerning

(*a*) The annulus or ring.

$$\pi R^2 - \pi r^2 = \pi(R + r)(R - r).$$

(*b*) Hollow cylinders or pipes.

$$\pi R^2h - \pi r^2h = \pi(R + r)(R - r)h.$$

Example: What is the weight of a 12-foot length of cast iron pipe 3 inches in external diameter, 1 inch thick, the weight of cast iron being $\frac{1}{4}$ lb. per cubic inch?

Solution:
$$22/7(18 + 17)(18 - 17)(144)(\tfrac{1}{4})$$
$$= 22/7 \times 35 \times 1 \times 36$$
$$= 110 \times 36$$
$$= 3{,}960 \text{ lbs.}$$

6. Polynomials by grouping, particularly

(*a*) $ax + ay + bx + by$, "two and two."

A binomial factor common to the two groups.

(*b*) $x^2 + 2ax + a^2 - y^2$, "three and one."

The difference of two squares, the first a trinomial and the other a monomial.

Types 6 (*a*) and (*b*) require facility in removing and inserting signs of aggregation, and show in a striking way their usefulness.

Note 1: It is suggested that the expressions to be factored should often be given in a haphazard order, needing rearrangement according to the powers of a letter.

Note 2: Before completing the drill on each type of factoring a brief review is advisable covering examples of the types previously studied, and simple two-step examples resulting in three or more factors. In first-year work it seems advisable to aid the pupil by stating with most examples of the latter sort the number of factors expected.

Note 3: The above scheme of factoring is comprehensive so far as concerns the product of any two binomials. For the rest the student may well be reminded that there is more to learn, in intermediate and college algebra.

Note 4: The following directions constitute a plan of attack for factoring any expression: First remove any common factors. Next count your terms.

If there are two terms is it the difference of two squares?

If there are three terms and you cannot factor it by inspection try the cross-product method.

If there are more than three terms try to group them "two and two," type 6*a*, or "three and one," type 6*b*.

(*Concluded in next issue*)

NEW BOOKS.

A Handbook of American Private Schools, 1919 Edition. By PORTER E. SARGENT. Boston. Pp. 761. Price $3.00.

Each year this Handbook improves in completeness, usefulness and authority. It is a book that no one having dealings with the private schools or summer camps of this country can afford to be without. It lists both schools and camps in easily accessible form, and gives their characteristics in a fairminded, comprehensive way that cannot fail to help parents as well as educators.

The discussion of formative movements, the bibliography, and other such information is invaluable to the school itself.

Applied Science for Wood-Workers. By WILLIAM H. DOOLEY. New York: The Ronald Press Co. Pp. viii + 457. Price $2.00.

Applied Science for Metal Workers. By the same author. Pp. viii + 479. Price $2.00.

The same principle underlies these two books, that of giving a practical understanding of general science, and building on that foundation the technical science needed for the trade in question. The texts are identical through the first 368 pages, after which each one takes up the applied science fitted to its title.

While these books are particularly well fitted for use in vocational or trade schools, they seem to present interesting possibilities for use in a general high-school course, for their practical handling of everyday problems, illustrated, as it is, with photographs and examples from many industries, gives a very concrete and interesting approach to science.

Number by Development. Vol. II., Intermediate Grades, 486 + xi. Vol. III., Grammar Grades, 514 + xx. By JOHN C. GRAY. Philadelphia: J. B. Lippincott Co.

These books not only discuss the principles applicable to number teaching, but give the details of the development of the important parts of the subject so that they furnish the teacher with a plan for the class development of new ideas and methods. Vol. II. takes up fractions, while Vol. III. is largely concerned with decimals, measurement, and the use of percentage and its applications.

The books are partly the result of experiment and trial in the schools of Clearfield, Massachusetts, of which Mr. Gray is Superintendent.

Junior High-School Mathematics, Third Course. By WILLIAM L. VOSBURGH and FREDERICK W. GENTLEMAN. New York: The Macmillan Company. Pp. viii + 295.

The first two books of this series were designed to fit the needs of the first and second years of the junior high school. The book is for the last year of junior high school, or the first year of an ordinary high school.

It includes some arithmetic, the elements of algebra with many of the complications omitted, and quite a good deal of geometry, including both measurement and some of the demonstrative geometry.

The book is consistent in the way it completes the course for the junior high school, but it does not seem to exactly fit the best thought on what should be taught in the first year of the ordinary high school.

Famous Generals of the War. By CHARLES H. L. JOHNSON. Boston: The Page Company. Pp. 310. $2.00 net.

The subjects treated in this volume are: Joffre, French, King Albert, Foch, Haig, Pershing, Petain, Diaz, Allenby, Maude, D'Esperey, De Castelnau, Smuts, Byng; and in the accounts given the great battles and crises of the war are woven so that one who reads the book will have a very good account of the war. The language is vivid and forceful and the illustrations are good.

Henley's Twentieth Century Book of Recipes, Formulas and Processes. New York: Norman W. Henley Publishing Co. $4.00.

This book of 800 pages is the most complete book of recipes ever published, giving thousands of receipes for the manufacture of valuable articles for every-day use. Hints, helps, practical ideas and secret processes are revealed within its pages. It covers every branch of the useful art.

The book to which you may turn with confidence that you will find what you are looking for. A mine of information, up-to-date in every respect. Contains an immense number of formulas that every one ought to have that are not found in any other work.

Threads and Thread Cutting. By COLVIN and STABEL. New York: Norman W. Henley Publishing Co. Third edition. 35 cents.

This clears up many of the mysteries of thread cutting, such as double and triple threads, internal threads, catching threads, use of hobs, etc. Contains a lot of useful hints and several tables.

House Wiring. By THOMAS W. POPPE. New York: Norman W. Henley Publishing Co. Third edition revised and enlarged. 125 pages, fully illustrated, flexible cloth. 75 cents.

Describing and illustrating up-to-date methods of installing electric light wiring. Contains just the information needed for successful wiring of a building. Fully illustrated with diagrams and plans. It solves all wiring problems and contains nothing that conflicts with the rulings of the National Board of Fire Underwriters.

NOTES AND NEWS.

THE Thirty-third Meeting of the Association of Teachers of Mathematics in the Middle States and Maryland was held Saturday, November 29, 1919, at the University of Pennsylvania, Philadelphia, Pa., in affiliation with the Asociation of Colleges and Preparatory Schools in the Middle States and Maryland. The morning and afternoon sessions were devoted to a discussion of the Preliminary Report of the Committee on Mathematical Requirements appointed by the Mathematical Association of America.

At the morning session Dean Herbert E. Hawkes of Columbia University gave an Exposition of Algebraic Section of the Report. His paper was followed by a Criticism and Discussion by Howard F. Hart, Head of the Department of Mathematics, Mont Clair High School. A general discussion followed which showed an enthusiastic acceptance of the trend of the report.

After a brief business meeting, the afternoon session was devoted to a consideration of the Geometric Section of the Report. An Exposition of this material was given by Mr. Raleigh Shorling of the Lincoln School, New York City. His paper was followed by a Criticism and Discussion by Mr. C. B. Walsh, Principal of the Friends' Central School, Philadelphia, Pa. A general discussion from the floor showed that the members present were in general sympathy with the trend of this report.

Fellow Teacher:

A preliminary canvass of the judgments of a number of active teachers of mathematics gives unmistakable assurance to the temporary committee of the Mathematics Club of Chicago of the advisability of initiating a National Council of Mathematics Teachers.

The organization meeting of the Council is set for February, 1920, at Cleveland, along with the meeting of the Department of Superintendence of the N. E. A. The temporary committee is arranging a live program for the meeting. The best men in the country will speak.

The exact nature and function of the Council will be formulated at the organization meeting. It seems probable that this national organization will serve to unify and vitalize and coordinate the work of the many separate and independent organizations throughout the country. Through its official organ, a monthly magazine, the best professional thought and leadership would be made available to the members of our profession.

You, as an official of your organization of mathematics teachers, are asked to attend to the appointment of at least (three or five) delegates who will represent you in the National Council meeting. Urge as many as possible to attend.

Will you please inform the committee when your organization has acted in this matter, sending along also the names of those appointed as delegates to the first meeting of the National Council.

Professionally yours,

W. W. Gorsline,
J. R. Clark,
M. J. Newell,
J. A. Foberg,
C. M. Austin,
Oak Park, Ill.,
Chairman.

Determination among the school leaders of Great Britain that the lessons of thrift and carefully living and spending gained during the war and since, shall not be lost, has led to the establishment of the continuation schools which will open formally in January.

These schools form a part of the educational reform now being effected in England as a result of post war conditions. The continuation schools are meant to bridge the gap between the time when the ordinary boy leaves school and the time when he settles down to a life vocation. Their influence on the life of the community, according to British officials will not lie merely in acquirement of knowledge. Behind these scholastic advantages will be others no less potent for the good of the nation.

The British House of Commons, in voting down recently the proposal that an issue of "lottery bonds" be made a part of that country's after-the-war financial scheme, has definitely aligned

Great Britain with the United States in the adoption of the "Work and Save" program as the only safe and sure path to financial restoration.

Austen Chamberlain, chancellor of the exchequer, in opposing the lottery, said, according to dispatches, that the "only salvation for the country" was for every man to settle down to hard work and steady saving. The Commons endorsed this position of the Government by a vote of 276 to 84.

THE fall meeting of the Southern Section of the Association of Teachers of Mathematics, was held at Goucher College, Baltimore, on December 23.

A report of the Philadelphia meeting was given by Mr. E. R. Smith, Park School, Baltimore. Prof. A. B. Cohen, of Johns Hopkins, spoke on the College courses for teachers in the summer schools. Miss Elizabeth White, of Eastern High School, Baltimore, gave a report of the work of the Mathematical Club in her school. At the afternoon session an address was given

on Commercial Mathematics in the High School, by Prof. W. S. Schlauch, of the High School of Commerce, New York City.

American School Boys and Girls Will Exchange Weekly Letters and Information with French.

Exchange of Historical, Geographical, Home-life, Commercial and Manufactural Material Planned—Also Kodak Views and Clippings.

With the approval of the Department of State and the United States Bureau of Education, and the co-operation of the French Ministry of Education, there will be operated in the United States, beginning with the school year, a National Bureau of French-American Education Correspondence, to be located at George Peabody College, Nashville, Tenn. The new bureau will promote correspondence between hundreds of thousands of pupils in France who are studying English and the pupils in America who are studying French.

The bureau will obtain from each teacher of French in the United States the list of pupils recommended for correspondence. Similar lists will be obtained from the teachers of English in France. For each pupil there will be given personal data as to age, sex, preparation, and main interests, so that the bureau may select the best-suited correspondents for each individual pupil.

Boys will correspond with boys, and girls with girls. From the bureau, teachers in America will receive a list of carefully selected French correspondents, so distributed in all the representative French and Belgian centers and the war area that there will be the maximum benefit for the class as a whole. For French and Belgian classes, there will be a similar representation of American centers.

The plan is that the French and American correspondents exchange weekly educational letters, each writing first in his own language and later in the language of his foreign correspondent. Linguistic training will not be the only educational end served. Along with the letters, there will be a fine exchange of historical, artistic, geographical, manufactural, commercial, and home-life material and information, clippings, picture pos-

tals, kodak views, etc., leading up to the deepest exchanges of human sympathies and ideals, that will reinforce international good will.

All the correspondence coming to the members of a given class will be kept on a bulletin board for the benefit of teacher and class. At general exercises in the schools, the foreign-language classes may present the most interesting phases of the correspondence to the entire school. The bureau will issue bulletins to the teachers, showing how to direct the pupils in this correspondence. Colleges and universities, private classes and clubs, as well as high schools, are included in the plan.

George Peabody College for teachers, Nashville, Tenn., will furnish the housing and general administration. It is planned, if funds permit, to establish within a few months, also, a Spanish-American bureau for the schools where Spanish is taught.

Through the co-operation of the French Ministry of Education all the schools, lycees, colleges, and universities of France are responding to the movement, so that many lists of French correspondents are already being received. Any institutions in America where French is taught or where there are students who can read French, as well as all private classes, clubs, or study circles, will be served by the bureau. Literature and enrollment blanks will be sent throughout the country. Any institutions or classes not otherwise reached may write to the bureau.

The Day After Christmas.

On the day after Christmas most people pick up the wrapping paper and string scattered about and ask themselves whether they really had as much joy and pleasure from the giving and receiving of gifts as they ought to have had. They know that those to whom they gave, liked and appreciated the spirit which led to the giving, but did they like the gifts? The overflowing ashbarrels, the crowd around the exchange counters in the stores and additional packages on the top shelves of unused closets and attics give the answer, in many cases.

It is less trouble and less effort to give sensible and useful and appropriate gifts than those which will bring neither profit nor gratification to the recipient. No matter for whom the gift is

designed nor how expensive or inexpensive you desire to make it, Government Savings Securities will supply the giver's needs and the receiver's wants. Thrift Stamps, War Savings Stamps, Treasury Savings Certificates and Liberty Bonds are adjusted to the limitations of every purse.

Moreover, their usefulness is not momentary. They will not be cast aside as out of fashion or outgrown, for they grow and increase in value with the passage of time. They carry with them the spirit of desire for future well-being expressive of true friendship and affection. They are appropriate for all and more than a little excuse exists for a recipient to feel that little thought has been spent on a gift no matter how expensive, if it is manifestly inappropriate. Buy W. S. S.

Archdeacon B. Talbot Rogers, former President of Racine College, Fond du Lac, Wisconsin, who has lately returned from the Balkans, gave some interesting impressions of Serbia and the Serbian at the offices of the Serbian Relief Committee of America, 70 Fifth Avenue, New York—William Jay Schieffelin, Chairman.

Dr. Rogers's party heard first-hand reports of the impossible conditions in the south. It was there that the Bulgarians were in control and the worst atrocities were committed, all forms of cruelties were suffered, of which teachers and preachers were the special victims. According to the cold Turkish idea, the country that can get its teachers into other lands becomes ruler of those lands. That was the belief held and acted upon by the Bulgarians. Teachers were not merely exterminated, they were killed in the most atrocious manner, tortured with the utmost fiendishness, thrown into pits to be buried alive; burned and crucified.

"There is no question about the need of relief in Serbia," Dr. Rogers asserted. "Through last winter the people existed on roots and grasses; all their food and farm implements were looted."

Since the visit of Dr. Rogers to Belgrade much has been accomplished through the unceasing efforts of the new government in co-operation with the great British and American Relief organizations. Absolute wonders have been worked in some

directions, yet in all the net results in comparison to the appalling need are woefully small. There seems to be a concerted effort at this critical time by means of press despatches dated from Belgrade and optimistic reports, to persuade the American people that Serbia no longer needs its help. Such despatches and reports either are based upon half-truths unconsciously distorted, or upon wilful misrepresentations by those who fear the recovery of the Serbian race,—fear it because it means one more vigorous and incorruptible guardian of the Gateway of the East.

The work of the Serbian Relief Committee of America lies among Serbia's helpless little children, who for four years have endured almost inconceivable privations and hardships. It is responsible for the preservation of forty thousand such children in the Chachak district alone, where Serbian headquarters of the Committee are established. The very continuance of a gallant race is dependent upon the lives of these men and women of the next generations, and, if there is any relaxation of effort at this critical time, thousands upon thousands of them must die before the opening of spring.

Scientific Food Selection Simplified.

The principles of scientific food selection for the average family have been reduced to easily applied terms in a series of six charts designed for popular use by teachers and lecturers. The charts, prepared by the Office of Home Economics, United States Department of Agriculture, are so arranged that the value in calories of any meal containing any of the several foods listed and the cost can be easily calculated.

In general, the tabulations are planned to show unchanging factors on which wise food selections must be based—food requirements and food composition—and to provide spaces for the changing factors—prices. With one or two exceptions the foods listed are those for which the Department of Labor quotes prices in its monthly reports.

These charts can be easily reproduced in either temporary or permanent form. For the former use they can be copied on a schoolroom blackboard. It they are desired for more permanent use it is suggested that they be made on blackboard cloth.

In this form they can be rolled compactly. The charts were used by a Department of Agriculture speaker to illustrate lectures at the recent National Dairy Show in Chicago.

The first chart shows in 100-calorie portions the amount of food needed daily by the average family and also the proportions of the wholesome diet; that is, the desirable relative amounts of the following five classes of foods. I., Vegetables and Fruits; II., Milk, Meat, Eggs, and Similar Foods; III., Cereal Foods; IV., Sweets; and V., Fats. Charts like this can be made more generally useful if the words "average family (father, mother, and three young children)" and also the number of 100-calorie portions are omitted. The speaker can then fill in these spaces and apply the chart to the needs of any individual or family, to a man at moderately active muscular work, for example, or to a family of two average adults. Most teachers have the necessary data for this. They should remember, however, that these charts are designed to show the amount of food that should be purchased rather than the amount to be eaten. The amounts inserted should therefore make provision for all the losses that take place in the course of storing the food and of preparing it for the table. An allowance of 10 per cent. for this is customary.

For the Working Man.

For illustration, it is generally agreed that a man who does moderately active muscular work needs about thirty 100-calorie portions daily. If he is to receive this amount it will be necessary to provide at least thirty-three 100-calorie portions unless extraordinary care is taken to prevent waste. With the proportions of a wholesome and attractive diet given on the chart it is a simple matter to estimate the number of 100-calorie portions needed of the various kinds of food. In the case of this man the distribution would be about as follows: From vegetables and fruits, 7; from meat, milk, eggs, etc., 8; from cereal foods, 10; from sweets, 3; and from fats, 5.

If the food for the average family costs on the average one cent per 100-calorie portion, the total expense for the day will of course be $1.20, exclusive of tea, coffee, spice, etc. If it costs, on the average 1.5 cents per 100-calorie portion, the expense per

day will be $1.80. If an effort is to be made to keep the expense down to $1.80 per day, for example, and if some foods cost 4 or 5 cents per 100-calorie portion, others must be considerably cheaper.

The other five charts, one for each group of foods, show how food materials that have somewhat the same uses in the diet can be quickly compared in price. Painted on each chart opposite the name of the food material is the number 100-calorie portions it provides per pound, per quart, or per dozen. In the next column is an empty space to be filled with the price, which of course fluctuates. This price can often be obtained from a member of the audience and inserted on the chart with chalk. The effect of this is to show the immediate practical usefulness of the information on the charts and to add interest to the discussion. From the price and from the number in the second column the price per 100-calorie portions can be quickly estimated. For example, the chart shows that medium-sized oranges furnish about ten 100-calorie portions per dozen. When oranges cost 40 cents a dozen, therefore, they furnish fuel at a cost of 4 cents per 100-calorie portion; when they cost 60 cents a dozen, they furnish fuel at the cost of 6 cents per 100-calorie portion. Raisins furnish about fourteen 100-calorie portions per pound. If they cost 28 cents a pound, they furnish fuel for 2 cents per 100-calorie portion.

How Costs Can Be Cut.

None of the representative foods listed in the first and second group furnish body fuel for less than 2 cents per 100-calorie portion, and many of them are much higher priced. It is upon the foods of the third, fourth, and fifth groups, particularly the third and fourth, that the housekeeper must depend to keep down the cost of the diet as a whole. For example, flour at 8 cents a pound furnishes body fuel for ½ cent a 100-calorie portion; corn meal at 5 cents a pound furnishes at at amout ⅓ of a cent a 100-calorie portion. There is of course a limit to the amount of these lower-priced foods that can be safely used. This limit is suggested by the first chart.

The proportions of the different food groups suggested as desirable to make up the total food fuel of the day's food are

not to be taken as an absolute rule. Experience has shown that they are reasonable, however, and that when they are followed the diet is likely to be wholesome and good tasting, and will supply the body in suitable proportions with the various food substances it needs.

Chart 1.

THE ADEQUATE DIET FOR THE "AVERAGE" FAMILY.

(Father, Mother, and Three Young Children.)

Provides daily about 120 one-hundred calorie portions, distributed somewhat as follows:

Vegetables and Fruits.................	24 one-hundred calorie portions (20 per cent. of the total)
Milk, Eggs, Meat, etc.................	36 one-hundred calorie portions (30 per cent. of the total)
Cereals	30 one-hundred calorie portions (25 per cent. of the total)
Sugar and Sugary Foods...............	12 one-hundred calorie portions (10 per cent. of the total)
Fats and Fat Foods....................	18 one-hundred calorie portions
	120

Chart 2.

GROUP 1.

FRUITS AND VEGETABLES.

(Depended on for bulk, flavor, minerals and vitamins.)

	100-Calorie Portions.	Price.	Price per 100-Calorie Portion.
Potatoes	3 per lb.	— cts. per lb.	— cts.
Onions..............	2 " "	" " "	"
Cabbage	1 " "	" " "	"
Corn, canned	5 " " No. 2 can	" " " No. 2 can	"
Peas, canned	3 " " " " "	" " " " " "	"
Tomatoes, canned....	1 " " " " "	" " " " " "	"
Prunes	11 " "	" " "	"
Raisins	14 " "	" " "	"
Oranges (8 oz. each)..	10 " doz.	" " doz.	"
Bananas (5 oz. each)..	11 " "	" " "	"

Chart 3.

GROUP 2.

MILK, EGGS, MEAT, AND SIMILAR FOODS.

(Depended on for efficient protein and fat.)

	100-Calorie Portions.	Price.	Price per 100-Calorie Portion.
Cheese	20 per lb.	—— cts. per lb.	—— cts.
Eggs	9 " doz.	" " doz.	"
Sirloin steak	10 " lb.	" " lb.	"
Round steak	7 " "	" " "	"
Rib roast	11 " "	" " "	"
Chuck roast	7 " "	" " "	"
Plate beef	12 " "	" " "	"
Pork chops	13 " "	" " "	"
Ham	15 " "	" " "	"
Lamb	11 " "	" " "	"
Hens	8 " "	" " "	"
Salmon, canned	7 " "	" " "	"
Mackerel, salt	11 " "	" " "	"
Oysters	5 " qt.	" " qt.	"
Milk*	6 " "	" " "	"

* Needed for growth.

Chart 4.

GROUP 3.

CEREAL FOODS AND DRIED LEGUMES.

(Depended on for protein and starch.)

	100-Calorie Portions.	Price.	Price per 100-Calorie Portion.
Corn meal	16 per lb.	— cts. per lb.	— cts.
Rolled Oats	18 " "	" " "	"
Wheat flour	16 " "	" " "	"
Bread	12 " "	" " "	"
Rice	16 " "	" " "	"
Macaroni	16 " "	" " "	"
Corn flakes	16 " "	" " "	"
Beans, dried	16 " "	" " "	"

Chart 5.

GROUP 4.

SUGAR AND SUGARY FOODS.

(Depended on for flavor and for body fuel.)

	100-Calorie Portions.	Price.	Price per 100-Calorie Portion.
Sugar, granulated	18 per lb.	— cts. per lb.	— cts.
Sugar, lump	18 " "	" " "	"
Sugar, maple	13 " "	" " "	"
Honey	15 " "	" " "	"
Molasses	13 " "	" " "	"
Sirup, corn	14 " "	" " "	"
Candy	17 " "	" " "	"

Chart 6.

GROUP 5.

FATS AND FAT FOODS.

(Depended on for richness and for body fuel.)

	100-Calorie Portions.	Price.	Price per 100-Calorie Portion.
Butter*	34 per lb.	— cts. per lb.	— cts.
Lard	41 " "	" " "	"
Vegetable oils	41 " "	" " "	"
Bacon	26 " "	" " "	"
Cream*	9 " qt.	" " qt.	"

* Important for vitamins, growth essentials.